JOURNAL OF FUNCTIONAL ANALYSIS **142**, 79–88 (1996)
ARTICLE NO. 0144

Invariant Submeans and Semigroups of Nonexpansive Mappings on Banach Spaces with Normal Structure*

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Received January 25, 1995; revised December 28, 1995

In this paper we study the relation between invariant submean and normal structure in a Banach space. This is used to give an improvement and different proof of a fixed point theorem of Lim (also of Belluce and Kirk for commutative semigroups) for left reversible semigroup of nonexpansive mappings on weakly compact convex subsets of a Banach space with normal structure. © 1996 Academic Press, Inc.

0. INTRODUCTION

Let S be a semitopological semigroup, i.e., S is a semigroup with Hausdorff topology such that for each $a \in S$, the mappings $s \to sa$ and $s \to as$ from S into S are continuous. Let $RUC(S)$ denote the space of bounded right uniformly continuous real-valued functions on S; S is called *left reversible* if any two closed right ideals of S have non-void intersection. A closed convex subset K of a Banach space E has *normal structure* [6, p. 39] if for each bounded closed convex subset H of K which contains more than one point, there is a point $x \in H$ which is not a diametral point of H, i.e., $\sup\{\|x-y\| : y \in H\} < \delta(H)$, where $\delta(H) =$ the diameter of H.

Belluce and Kirk [3] first proved that if K is a nonempty weakly compact convex subset of a Banach space and if K has complete normal structure, then every family of commuting nonexpansive self-maps on K has a common fixed point. Later Lim [14, Theorem 3] extended this theorem to

* This research is supported by an NSERC grant and IBMJAPAN, Ltd.

0022-1236/96 $18.00

a continuous representation of a left reversible semitopological semigroup S as nonexpansive mappings on a weakly compact convex set K with normal structure.

In this paper, we shall prove (Theorem 3.1), among other things, that Lim's result remains valid when $RUC(S)$ has a left subinvariant submean μ, i.e., $\mu(l_a f) \geqslant \mu(f)$ for all $f \in RUC(S)$ and $a \in S$, where $(l_a f)(s) = f(as)$, $s \in S$. Note that if S is left reversible, then $CB(S)$, the space of bounded continuous real-valued functions on S, has a left subinvariant submean (see the proof of Corollary 3.2). However S need not be left reversible even when $CB(S)$ has a left invariant mean (see [9]). Our result answers affirmatively a problem posed during the Conference on Fixed Point Theory and Applications held at CIRM, Marseille–Luminy, 1989 (see [11, p. 307, Problem 5]).

The notion of submean was first introduced by Mizoguchi and the second author in [15]. It turns out to be an effective notion in non-linear fixed point and ergodic theories (see [12], [17], and [18]). We note that Lim's proof of his fixed point theorem relies heavily on the order structure on the semigroup S defined by left reversibility (i.e., $a \preccurlyeq b$ if and only if $\overline{aS} \supseteq \overline{bS}$). Our proof of Theorem 3.1 depends on rather careful analysis in the relation of submean and normal structure in a Banach space (see Section 2). It also depends on Fan's theorem for convex inequalities [5].

1. SOME PRELIMINARIES

All topologies in this paper are assumed to be Hausdorff. If E is a Banach space and E^* is its continuous dual, then the value of $f \in E^*$ at $x \in E$ will be denoted by $f(x)$ or $\langle f, x \rangle$. Also if $A \subseteq E$, then $\overline{co}\, A$ will denote the closed convex hull of A in E.

Given a non-empty set S, we denote by $l^\infty(S)$ the Banach space of bounded real-valued functions on S with the supremum norm.

Let X be a closed subspace of $l^\infty(S)$ containing constants. Then $\mu \in X^*$ is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. As is well known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leqslant \mu(f) \leqslant \sup_{s \in S} f(s)$$

for each $f \in X$.

By a *submean* on X, we shall mean a real-valued function μ on X satisfying the following properties:

(1) $\mu(f+g) \leqslant \mu(f) + \mu(g)$ for every $f, g \in X$;

(2) $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geqslant 0$;

(3) for f, $g \in X$, $f \leqslant q$ implies $\mu(f) \leqslant \mu(g)$;

(4) $\mu(c) = c$ for every constant function c.

The value of a submean μ on X at f will also be denoted by $\langle \mu, f \rangle$ or $\mu_t(f(t))$.

Let S be a semigroup. Then a subspace X of $l^\infty(S)$ is *left* (resp. *right*) *translation invariant* if $l_a(X) \subseteq X$ (resp. $r_a(X) \subseteq X$) for all $a \in S$, where $(l_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$, $s \in S$. If S is a semitopological semigroup, we denote by $CB(S)$ the closed subalgebra of $l^\infty(S)$ consisting of continuous functions. Let $LUC(S)$ (resp. $RUC(S)$) be the space of *left* (resp. *right*) *uniformly continuous functions* on S, i.e., all $f \in CB(S)$ such that the mapping from S into $CB(S)$ defined by $s \to l_s f$ (resp. $s \to r_s f$), is continuous when $CB(S)$ has the sup norm topology. Then as is known [4] $LUC(S)$ and $RUC(S)$ are left and right translation invariant closed subalgebras of $CB(S)$ containing constants. Note that when S is a topological group, then $LUC(S)$ is precisely the space of right uniformly continuous functions on S defined in [8].

If S is a semitopological semigroup, and C is a nonempty subset of a Banach space E, then a representation $\mathcal{S} = \{T_s; s \in S\}$ of S as mappings from C into C is *continuous* if the map $S \times C \to C$ defined by $(s, x) \to T_s x$, $s \in S$, $x \in C$, is continuous when $S \times C$ has the product topology.

The following Lemma of Lim [13] will be useful for us:

LEMMA 1.1. *A closed convex subset C of a Banach space has normal structure if and only if it does not contain a sequence $\{x_n\}$ such that for some $c > 0$, $\|x_n - x_m\| \leqslant c$, $\|x_{n+1} - \overline{x_n}\| \geqslant c - (1/n^2)$ for all $n \geqslant 1$, $m \geqslant 1$, where $\overline{x_n} = (1/n) \sum_{i=1}^n x_i$.*

2. NORMAL STRUCTURE AND SUBMEAN

In this section, we shall establish some properties relating the concepts of normal structure and of submean.

Let S be a non-empty set and f be a function from S into C such that $\{f(s); s \in S\}$ is bounded, where C is a closed convex subset of a Banach space E with more than one point. For each $x \in C$, define $f_x(s) = \|f(s) - x\|$, $s \in S$. Let X be a closed subspace of $l^\infty(S)$ containing constants such that $f_x \in X$ for each $x \in X$ and let μ be a submean on X. Define $r: C \to R$ by $r(x) = \langle \mu, f_x \rangle$.

LEMMA 2.1. *The function $r: C \to R$ is continuous, convex on C and if $\|x_n\| \to \infty$, then $r(x_n) \to \infty$.*

Proof. Let $x_n \to x$, and x_n, x be in C. Then since

$$\|f(t)-x_n\| - \|f(t)-x\| \leqslant \|x_n - x\|$$

and

$$\|f(t)-x\| - \|f(t)-x_n\| \leqslant \|x_n - x\|,$$

it follows that

$$\langle \mu, f_{x_n} \rangle \leqslant \|x_n - x\| + \langle \mu, f_x \rangle$$

and

$$\langle \mu, f_x \rangle \leqslant \|x_n - x\| + \langle \mu, f_{x_n} \rangle.$$

Hence

$$|\langle \mu, f_{x_n} \rangle - \langle \mu, f_x \rangle| \leqslant \|x_n - x\| \to 0.$$

This implies that r is continuous.

If $\alpha, \beta \leqslant 0$, $\alpha + \beta = 1$ and $x, y \in C$, then since

$$\|f(t) - (\alpha x + \beta y)\| \leqslant \alpha \, \|f(t) - x\| + \beta \, \|f(t) - y\|,$$

we have

$$r(\alpha x + \beta y) \leqslant \alpha r(x) + \beta r(y).$$

Finally, if $\|x_n\| \to \infty$, then since $\|x_n\| \leqslant \|x_n - f(t)\| + \|f(t)\|$ for each $t \in S$, we have

$$\|x_n\| \leqslant r(x_n) + M \to \infty, \qquad \text{where} \quad M = \sup_{t \in S} \{ \|f(t)\| \}. \quad \blacksquare$$

Let $\rho_0 = \inf\{r(x);\, x \in C\}$. Then we have the following:

LEMMA 2.2. *If C is non-empty weakly compact and convex, then the set $A = \{x;\, r(x) = \rho_0\}$ is non-empty, closed, and convex. Furthermore, if $C = A$, then*

$$\rho_0 = \inf_{y \in C} \sup_{x \in C} \{ \|x - y\| \}.$$

Proof. Since $r\colon C \to R$ is continuous and convex, r is weakly lower semicontinuous. So, A is non-empty, closed and convex. Suppose $C = A$, let

$\varepsilon > 0$ and $\sigma = \{x_1, ..., x_n\} \subseteq C$. Consider a system $\{h_i\}_{i=1}^n$ of convex weakly lower semicontinuous functions on C defined by

$$h_i(x) = \|x - x_i\|, \qquad \text{for} \quad x \in C \qquad \text{and} \quad i = 1, ..., n.$$

We shall show that the set $D_\sigma = \{z \in C; h_i(z) \leqslant \rho_0 + \varepsilon, i = 1, ..., n\}$ is non-empty. In fact, from $A = C$, we have $r(x_i) = \rho_0$ for $i = 1, 2, ..., n$. So for any $\{\alpha_i\}_{i=1}^n \subseteq R$ with $\alpha_1, \alpha_2, ..., \alpha_n \geqslant 0$ and $\sum_{i=1}^n \alpha_i = 1$, we have

$$\mu_t\left(\sum_{i=1}^n \alpha_i \|f(t) - x_i\|\right) \leqslant \sum_{i=1}^n \alpha_i r(x_i) = \rho_0 < \rho_0 + \varepsilon.$$

Then, there exists $t_0 \in S$ such that

$$\sum_{i=1}^n \alpha_i \|f(t_0) - x_i\| \leqslant \rho_0 + \varepsilon$$

and hence $\sum_{i=1}^n \alpha_i h_i(f(t_0)) \leqslant \rho_0 + \varepsilon$. Hence by Fan's Theorem for convex inequalities on a topological vector space [5], there exists $z \in C$ such that

$$h_i(z) = \|z - x_i\| \leqslant \rho_0 + \varepsilon \qquad \text{for each} \quad i = 1, ..., n.$$

This implies that the set D_σ is non-empty. Hence by weak compactness of C, the set $D = \{z \in C; \|x - z\| \leqslant \rho_0 + \varepsilon \text{ for all } x \in C\}$ is also non-empty. So we have

$$\begin{aligned} \rho_0 &= \min\{r(y); y \in C\} \\ &\leqslant \min_{y \in C}\{\sup_{t \in S} \|f(t) - y\|\} \\ &\leqslant \min_{y \in C}\{\sup_{x \in C} \|x - y\|\} \leqslant \rho_0 + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\rho_0 = \inf_{y \in C}\{\sup_{x \in C} \|x - y\|\}. \quad \blacksquare$$

If S is a semigroup and X is left translation invariant, a submean μ on X is left *subinvariant* if $\mu(l_a f) \geqslant \mu(f)$ for each $f \in X$ and $a \in S$. A representation $\mathscr{S} = \{T_s; s \in S\}$ as mappings from a subset C of a Banach space into C is called *X-admissible* if for each $x, y \in C$, the function $t \to \|T_t x - y\|$ belongs to X.

THEOREM 2.3. *Let C be a non-empty weakly compact convex subset of a Banach space E. If C has more than one point and normal structure, then C satisfies*:

(P) *Whenever S is a semigroup, Φ is a closed left translation invariant subspace of $l^\infty(S)$ containing constants with a left subinvariant submean μ, $\mathcal{S} = \{T_s;\ s \in S\}$ is a Φ-admissible representation of S as non-expansive mappings from C into C, then the set $A_x = \{y \in C; \mu_t \|T_t x - y\| = \rho_x\}$ is a* proper *subset of C for some $x \in C$, where $\rho_x = \inf\{\mu_t \|T_t x - y\|; y \in C\}$. Furthermore, for each $x \in C$ the set A_x is non-empty, closed, convex, and T_s-invariant.*

Proof. If $A_x = C$ for all $x \in C$, then by Lemma 2.2,

$$\rho_x = \rho_0 = \inf_{x \in C} \{\sup_{y \in C} \|x - y\|\} = \mu_t \|T_t x - y\| \tag{1}$$

for all x, $y \in C$. So let

$$A_0 = \{z \in C; \sup_{t \in S} \|T_t x - z\| \leqslant \rho_0, \forall x \in C\}.$$

By weak compactness of C and (1), there exists $z_0 \in C$ such that

$$\sup_{y \in C} \|z_0 - y\| = \rho_0.$$

Hence A_0 is a non-empty set. Let $z_0 \in A_0$ and $s \in S$. Then for any $x \in C$,

$$\rho_0 = \mu_t \|T_t x - z_0\| \leqslant \mu_t \|T_{st} x - z_0\| \leqslant \sup_{t \in S} \|T_{st} x - z_0\|$$

$$\leqslant \sup_{t \in S} \|T_t x - z_0\| \leqslant \rho_0$$

and

$$\rho_0 = \mu_t \|T_t x - T_s z_0\| \leqslant \mu_t \|T_{st} x - T_s z_0\| \leqslant \sup_{t \in S} \|T_{st} x - T_s z_0\|$$

$$\leqslant \sup_{t \in S} \|T_t x - z_0\| \leqslant \rho_0$$

by left subinvariance of μ. For using Lim's Lemma (Lemma 1.1), fix $z_0 \in A_0$. Then since $\rho_0 = \mu_t \|T_t z_0 - z_0\|$, there exists $s_1 \in S$ such that $\|T_{s_1} z_0 - z_0\| \geqslant \rho_0 - 1$. Let $x_1 = z_0$, $x_2 = T_{s_1} z_0$ and

$$\overline{x_2} = \tfrac{1}{2} z_0 + \tfrac{1}{2} T_{s_1} z_0.$$

Since $\rho_0 = \mu_t \|T_t z_0 - \overline{x_2}\| \leqslant \mu_t \|T_{s_1 t} z_0 - \overline{x_2}\|$, there exists $s_2 \in S$ such that $\|T_{s_1 s_2} z_0 - \overline{x_2}\| \geqslant \rho_0 - (1/2^2)$. So, let $x_3 = T_{s_1 s_2} z_0$. Then, we have

$$\|x_1 - x_2\| = \|z_0 - T_{s_1} z_0\| \leqslant \sup_{t \in S} \|z_0 - T_t z_0\| = \rho_0,$$

$$\|x_2 - x_3\| = \|T_{s_1} z_0 - T_{s_1 s_2} z_0\| \leqslant \|z_0 - T_{s_1} z_0\| \leqslant \rho_0,$$

and

$$\|x_3 - x_1\| = \|T_{s_1 s_2} z_0 - z_0\| \leqslant \sup_{t \in S} \|T_t z_0 - z_0\| = \rho_0.$$

Similarly, let

$$\overline{x_3} = \tfrac{1}{3} x_1 + \tfrac{1}{3} x_2 + \tfrac{1}{3} x_3.$$

Then $\rho_0 = \mu_t \|T_t z_0 - \overline{x_3}\| \leqslant \mu_t \|T_{s_1 s_2 t} z_0 - \overline{x_3}\|$, there exists $s_3 \in S$ such that $\|T_{s_1 s_2 s_3} z_0 - \overline{x_3}\| \geqslant \rho_0 - (1/3^2)$. So, let $x_4 = T_{s_1 s_2 s_3} z_0$. Then, we have

$$\|x_4 - x_1\| = \|T_{s_1 s_2 s_3} z_0 - z_0\| \leqslant \sup_{t \in S} \|T_t z_0 - z_0\| = \rho_0,$$

$$\|x_4 - x_2\| = \|T_{s_1 s_2 s_3} z_0 - T_{s_1} z_0\| \leqslant \|T_{s_2 s_3} z_0 - z_0\| \leqslant \sup_{t \in S} \|T_t z_0 - z_0\| = \rho_0,$$

and

$$\|x_4 - x_3\| = \|T_{s_1 s_2 s_3} z_0 - T_{s_1 s_2} z_0\| \leqslant \|T_{s_3} z_0 - z_0\| \leqslant \sup_{t \in S} \|T_t z_0 - z_0\| = \rho_0.$$

By mathematical induction, let $x_5 = T_{s_1 s_2 s_3 s_4} z_0$, $x_6 = T_{s_1 s_2 s_3 s_4 s_5} z_0$, ... Then we have

$$\|x_n - x_m\| \leqslant \rho_0, \quad \forall n, m \qquad \text{and} \qquad \|x_{n+1} - \overline{x_n}\| \geqslant \rho_0 - \frac{1}{n^2}.$$

Using Lim's Lemma, C does not have normal structure. This is a contradiction. To see that A_x is T_s-invariant, $x \in C$, $s \in S$, let $y \in A_x$. Then

$$\rho_x \leqslant \mu_t(\|T_t x - T_s y\|) \leqslant \mu_t(\|T_{st} x - T_s y\|) \leqslant \mu_t(\|T_t x - y\|) = \rho_x.$$

Hence

$$\mu_t(\|T_t x - T_s y\|) = \rho_x, \qquad \text{i.e.,} \quad T_s y \in A_x. \quad \blacksquare$$

PROPOSITION 2.4. *Let K be a closed convex subset of a Banach space E. If K satisfies the following property* (Q), *then K has normal structure*:

(Q) *Whenever S is a non-empty set and f is a function from S into C, where C is a bounded closed convex subset of K with more than one point, and for each $x \in C$, the function $f_x(s) = \|f(s) - x\|$ belongs to a closed* *subspace X of $l^\infty(S)$ containing constants, and μ be a submean on X, $r(x) = \langle \mu, f_x \rangle$, $\rho_0 = \inf\{r(x); x \in C\}$, then the set $A = \{x \in C; r(s) = \rho_0\}$ is a proper subset of C.*

Proof. Suppose K does not have normal structure. Then there exists a bounded closed convex subset C of K such that $\delta(C) > 0$ and $\sup_{y \in C} \|y - x\| = \delta(C)$ for every $x \in C$. Let $S = C$. $X = l^\infty(S)$, $f\colon C \to C$, $f(s) = s$. Then for each $x \in C$, $f_x(s) = \|s - x\|$, $s \in C$ is in $l^\infty(C)$ (since C is bounded).

Let $\mu(h) = \sup h$, $h \in l^\infty(C)$, and $r(x) = \langle \mu, f_x \rangle$, $x \in C$, $r = \inf\{r(x); x \in C\}$. Then since $r(x) = \delta(C)$ for every $x \in C$, we have $r = \delta(C)$, i.e., $A = \{x \in C\colon r(x) = r\} = C$. This completes the proof. ∎

3. FIXED POINT THEOREMS

In this section we shall obtain a generalization of Lim's fixed point theorem [14] for left reversible semigroups of non-expansive mappings.

THEOREM 3.1. *Let S be a semitopological semigroup, let D be a non-empty weakly compact convex subset of a Banach space E which has normal structure and let $\mathscr{S} = \{T_s; s \in S\}$ be a continuous representation of S as non-expansive self mappings on D. Suppose $RUC(S)$ has a left subinvariant submean. Then $\mathscr{S}$ has a common fixed point in D.*

Proof. We first prove that for any $x \in D$ and $y \in E$, a function h defined by $h(t) = \|T_t x - y\|$ for all $t \in S$ is in $RUC(S)$. In fact, we have, for $s, u \in S$,

$$\begin{aligned}\|r_s h - r_u h\| &= \sup_{t \in S} |(r_s h)(t) - (r_u h)(t)| = \sup_{t \in S} |h(ts) - h(tu)| \\ &= \sup_{t \in S} |\, \|T_{ts}x - y\| - \|T_{tu}x - y\| \,| \leqslant \sup_{t \in S} \|T_{ts}x - T_{tu}x\| \\ &\leqslant \|T_s x - T_u x\|.\end{aligned}$$

Let

$$U = \{K \subset D\colon K \text{ is nonempty, closed, convex, and } T_s\text{-invariant}\}.$$

Then by Zorn's Lemma, there exists a minimal element C of U. Let $\delta(C) > 0$ and let μ be a left subinvariant submean. Then, for any $x \in C$,

$$A_x = \{z \in C\colon \mu_t \|T_t x - z\| = \min_{y \in C} \mu_t \|T_t x - y\|\}$$

is nonempty, closed, convex, and T_s-invariant (see Theorem 2.3). So, we have $A_x = C$ by minimality of C. Hence by Theorem 2.3, C cannot have normal structure which is a contradiction. ∎

COROLLARY 3.2. [14]. *Let S be a left reversible semitopological semigroup. Let D be a nonempty weakly compact convex subset of a Banach space E which has normal structure and let $\mathscr{S} = \{T_s; s \in S\}$ be a continuous representation of S as nonexpansive self mappings on D. Then $\mathscr{S}$ has a fixed point in D.*

Proof. If S is left reversible, define $\mu(f) = \inf_s \sup_{t \in sS} f(t)$. Then the proof of Lemma 3.6 in [12] shows that μ is a submean on $CB(S)$ such that $\mu(l_a f) \geqslant \mu(f)$ for all $f \in CB(S)$ and $a \in S$, i.e., μ is left subinvariant. ∎

EXAMPLE. E. Hewitt [7] has constructed a regular Hausdorff space T such that the only real continuous functions on T are the constant functions. Let $(U, \circ)$ be any discrete left amenable semigroup (e.g., $U =$ a commutative semigroup). Let $S = T \times U$ with multiplication defined by

$$(t_1, u_1) \cdot (t_2, u_2) = (t_1, u_1 \circ u_2),$$

with $t_1, t_2 \in T$, $u_1, u_2 \in U$ and product topology. Then S is a semitopological semigroup which is not left reversible since $(t, u) \cdot S = \{(t, u \circ u'); u' \in U\}$ is a closed right ideal of S, and $(t_1, u) \cdot S \cap (t_2, u) \cdot S = \phi$ if $t_1 \neq t_2$. However, if ϕ is a left invariant mean on $l^\infty(U)$ and $t_0 \in T$, then the positive functional m on $CB(S)$ defined by

$$m(f) = \phi(f_{t_0}) \qquad \text{for every} \quad f \in CB(S)$$

is a left invariant mean, where $f_{t_0}(u) = f(t_0, u)$ for every $u \in U$. In particular, for any such semitopological semigroup S, our fixed point theorem, Theorem 3.1, applies but the fixed point theorem of Lim [14] for left reversible semigroup (Corollary 3.2) does not (see also [9]).

Remark 3.3. It has been proved by Hsu [10] (see also [2]) that if S is discrete and left reversible, and $\mathscr{S} = \{T_s; s \in S\}$ is a representation of S as weakly continuous nonexpansive mappings on a weakly compact convex subset C of a Banach space, then C has a common fixed point for $\mathscr{S}$. Note that it follows from Alspach's example [1] that there exists a commutative semigroup of nonexpansive mappings on a weakly compact convex subset of $L_1[0, 1]$ with no common fixed point. But as is well known, $l_\infty(S)$ always has an invariant mean when S is commutative. Also Schechtman [16] has shown that there exists a weakly compact convex subset W of $L_1[0, 1]$ and a sequence $T_1, T_2, \ldots$ of commuting nonexpansive operators of W into itself such that any finite number of them have a common fixed point but there is no common fixed point for the entire sequence.

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